EBAMRGodunov

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Contents

1 Introduction

This document describes our numerical method for integrating systems of conservation laws (e.g., the Euler equations of gas dynamics) on an AMR grid hierarchy with embedded boundaries. We use an unsplit, second-order Godunov method, extending the algorithms developed by Colella [Col90] and Saltzman [Sal94].

2 Notation

All these operations take place in a very similar context to that presented in [CGL$^+$00]. For non-embedded boundary notation, refer to that document.

The standard (i, j, k) is not sufficient here to denote a computational cell as there can be multiple VoFs per cell. We define $\boldsymbol{v}$ to be the notation for a VoF and $\boldsymbol{f}$ to be a face. The function $ind(\boldsymbol{v})$ produces the cell which the VoF lives in. We define $\boldsymbol{v}^+(\boldsymbol{f})$ to be the VoF on the high side of face $\boldsymbol{f}$; $\boldsymbol{v}^-(\boldsymbol{f})$ is the VoF on the low side of face $\boldsymbol{f}$; $\boldsymbol{f}_d^+(\boldsymbol{v})$ is the set of faces on the high side of VoF $\boldsymbol{v}$; $\boldsymbol{f}_d^-(\boldsymbol{v})$ is the set of faces on the low side of VoF $\boldsymbol{v}$, where $d \in \{x, y, z\}$ is a coordinate direction (the number of directions is D). Also, we compose these operators to represent the set of VoFs directly connected to a given VoF: $\boldsymbol{v}_d^+(\boldsymbol{v}) = \boldsymbol{v}^+(\boldsymbol{f}_d^+(\boldsymbol{v}))$ and $\boldsymbol{v}_d^-(\boldsymbol{v}) = \boldsymbol{v}^-(\boldsymbol{f}_d^-(\boldsymbol{v}))$. The $\ll$ operator shifts data in the direction of the right hand argument. The shift operator can yield multiple VoFs. In this case, the shift operator includes averaging the values at the shifted-to VoFs.

We follow the same approach in the EB case in defining multilevel data and operators as we did for ordinary AMR. Given an AMR mesh hierarchy $\{\Omega^l\}_{l=0}^{lmax}$, we define the valid VoFs on level l to be

$$\mathcal{V}^l_{valid} = ind^{-1}(\Omega^l_{valid}) \tag{1}$$

and composite cell-centered data

$$\varphi^{comp} = \{\varphi^{l,valid}\}_{l=0}^{lmax}, \varphi^{l,valid} : \mathcal{V}^l_{valid} \to \mathbb{R}^m \tag{2}$$

For face-centered data,

$$\begin{aligned} \mathcal{F}^{l,d}_{valid} &= ind^{-1}(\Omega^{l,\mathbf{e}^d}_{valid}) \\ \vec{F}^{l,valid} &= (F_0^{l,valid}, \dots, F_{D-1}^{l,valid}) \\ F_d^{l,valid} &: \mathcal{F}^{l,d}_{valid} \to \mathbb{R}^m \end{aligned} \tag{3}$$

For computations at cell centers the notation

$$CC = A \mid B \mid C$$

means that the 3-point formula A is used for CC if all cell centered values it uses are available, the 2-point formula B is used if current cell borders the high side of the physical domain (i.e., no high side value), and the 2-point formula C is used if current cell borders the low side of the physical domain (i.e., no low side value). A value is "available" if its VoF is not covered and is within the domain of computation. For computations at face centers the analogous notation

$$FC = A \mid B \mid C$$

means that the 2-point formula A is used for FC if all cell centered values it uses are available, the 1-point formula B is used if current face coincides with the high side of the physical domain (i.e., no high side value), and the 1-point formula C is used if current face coincided with the low side of the physical domain (i.e., no low side value).

3 Equations of Motion

We are solving a hyperbolic system of equations of the form

$$\frac{\partial U}{\partial t} + \sum_{d=0}^{\mathbf{D}-1} \frac{\partial F^d}{\partial x^d} = S \tag{4}$$

For 3D polytropic gas dynamics,

$$\begin{aligned} U &= (\rho, \rho u_x, \rho u_y, \rho u_z, \rho E)^T \\ F^x &= \left(\rho u_x, \rho u_x^2, \rho u_x u_y, \rho u_x u_z, \rho u_x E + u_x p\right)^T \\ F^y &= \left(\rho u_y, \rho u_x u_y, \rho u_y^2, \rho u_y u_z, \rho u_y E + u_y p\right)^T \\ F^z &= \left(\rho u_z, \rho u_x u_z, \rho u_z u_y, \rho u_z^2, \rho u_z E + u_z p\right)^T \\ E &= \frac{\gamma p}{(\gamma - 1)\rho} + \frac{|\vec{u}|^2}{2} \end{aligned} \tag{5}$$

We are given boundary conditions on faces at the boundary of the domain and on the embedded boundary. We also assume there may be a change of variables $W = W(U)$ ($W \equiv$ "primitive variables") that can be applied to simplify the calculation of the characteristic structure of the equations. This leads to a similar system of equations in W.

$$\begin{aligned} \frac{\partial W}{\partial t} + \sum_{d=0}^{\mathbf{D}-1} A^d(W) \, \frac{\partial W^d}{\partial x^d} &= S' \\ A^d &= \nabla_U W \cdot \nabla_U F^d \cdot \nabla_W U \\ S' &= \nabla_U W \cdot S \end{aligned} \tag{6}$$

For 3D polytropic gas dynamics,

$$W = (\rho, u_x, u_y, u_z, p)^T$$

$$A^x = \begin{pmatrix} u_x & \rho & 0 & 0 & 0 \\ 0 & u_x & 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & u_x & 0 & 0 \\ 0 & 0 & 0 & u_x & 0 \\ 0 & \rho c^2 & 0 & 0 & u_x \end{pmatrix}$$

$$A^y = \begin{pmatrix} u_y & 0 & \rho & 0 & 0 \\ 0 & u_y & 0 & 0 & 0 \\ 0 & 0 & u_y & 0 & \frac{1}{\rho} \\ 0 & 0 & 0 & u_y & 0 \\ 0 & 0 & \rho c^2 & 0 & u_y \end{pmatrix}$$

$$A^z = \begin{pmatrix} u_z & 0 & 0 & \rho & 0 \\ 0 & u_z & 0 & 0 & 0 \\ 0 & 0 & u_z & 0 & 0 \\ 0 & 0 & 0 & u_z & \frac{1}{\rho} \\ 0 & 0 & 0 & \rho c^2 & u_z \end{pmatrix}$$

4 Approximations to $\nabla \cdot F$.

To obtain a second-order approximation of the flux divergence in conservative form, first we must interpolate the flux to the face centroid. In two dimensions, this interpolation takes the form

$$\widetilde{F}_{\boldsymbol{f}}^{n+\frac{1}{2}} = F_{\boldsymbol{f}}^{n+\frac{1}{2}} + |\bar{x}|(F_{\boldsymbol{f} \ll sign(\bar{x})\boldsymbol{e}^d}^{n+\frac{1}{2}} - F_{\boldsymbol{f}}^{n+\frac{1}{2}}) \tag{7}$$

where $\bar{x}$ is the centroid in the direction d perpendicular to the face normal. In three dimensions, define $(\bar{x}, \bar{y})$ to be the coordinates of the centroid in the plane (d^1, d^2) perpendicular to the face normal.

$$\widetilde{F}_{\boldsymbol{f}}^{n+\frac{1}{2}} = F_{\boldsymbol{f}}^{n+\frac{1}{2}}(1 - \bar{x}\bar{y} + |\bar{x}\bar{y}|)+ \tag{8}$$

$$F_{\boldsymbol{f} \ll sign(\bar{x})\boldsymbol{e}^{d^1}}^{n+\frac{1}{2}}(|\bar{x}| - |\bar{x}\bar{y}|)+ \tag{9}$$

$$F_{\boldsymbol{f} \ll sign(\bar{x})\boldsymbol{e}^{d^2}}^{n+\frac{1}{2}}(|\bar{y}| - |\bar{x}\bar{y}|)+ \tag{10}$$

$$F_{\boldsymbol{f} << sign(\bar{x})\boldsymbol{e}^{d^1} << sign(\bar{x})\boldsymbol{e}^{d^2}}^{n+\frac{1}{2}}(|\bar{x}\bar{y}|) \tag{11}$$

Centroids in any dimension are normalized by Δx and centered at the cell center. This interpolation is only done if the shifts that are used in the interpolation are uniquely-defined and single-valued.

We then define the conservative divergence approximation.

$$\nabla \cdot \vec{F} \equiv (D \cdot \vec{F})^c = \frac{1}{k_{\boldsymbol{v}} h}((\sum_{d=0}^{\mathbf{D}-1} \sum_{\pm=+,-} \sum_{\boldsymbol{f} \in \mathcal{F}_{\boldsymbol{v}}^{d,\pm}} \pm \alpha_{\boldsymbol{f}} \widetilde{F}_{\boldsymbol{f}}^{n+\frac{1}{2}}) + \alpha_{\boldsymbol{v}}^B F_{\boldsymbol{v}}^{B,n+\frac{1}{2}}) \tag{12}$$

The non-conservative divergence approximation is defined below.

$$\nabla \cdot \vec{F} = (D \cdot \vec{F})^{NC} = \frac{1}{h} \sum_{\pm=+,-} \sum_{d=0}^{\mathbf{D}-1} \pm \bar{F}^{n+\frac{1}{2}}_{\boldsymbol{v},\pm,d} \tag{13}$$

$$\bar{F}^{n+\frac{1}{2}}_{\boldsymbol{v},\pm,d} = \begin{cases} \frac{1}{N(\mathcal{F}^{d,\pm}_{\boldsymbol{v}})} \sum_{\boldsymbol{f} \in \mathcal{F}^{d,\pm}_{\boldsymbol{v}}} F^{n+\frac{1}{2}}_{\boldsymbol{f}} & \text{if } N(\mathcal{F}^{d,\pm}_{\boldsymbol{v}}) > 0 \\ F^{\text{covered},n+\frac{1}{2}}_{\boldsymbol{v},\pm,d} & \text{otherwise} \end{cases} \tag{14}$$

The preliminary update update of the solution of the solution takes the form:

$$U^{n-1}_{\boldsymbol{v}} = U^n_{\boldsymbol{v}} - \Delta t((1-k_{\boldsymbol{v}})(D \cdot \vec{F})^{NC}_{\boldsymbol{v}} + k_{\boldsymbol{v}}(D \cdot \vec{F})^c_{\boldsymbol{v}}) \tag{15}$$

$$\delta M = -\Delta t k_{\boldsymbol{v}}(1-k_{\boldsymbol{v}})((D \cdot \vec{F})^c_{\boldsymbol{v}} - (D \cdot \vec{F})^{NC}_{\boldsymbol{v}}) \tag{16}$$

δM is the total mass increment that has been unaccounted for in the preliminary update. See the EBAMRTools document for how this mass gets redistributed in an AMR context. On a single level, the redistribution takes the following form:

$$U^{n+1}_{\boldsymbol{v}'} := U^{n+1}_{\boldsymbol{v}'} + w_{\boldsymbol{v},\boldsymbol{v}'}, \delta M_{\boldsymbol{v}} \tag{17}$$

$$\boldsymbol{v}' \in \mathcal{N}(\boldsymbol{v}), \tag{18}$$

where $\mathcal{N}(\boldsymbol{v})$ is the set of VoFs that can be connected to $\boldsymbol{v}$ with a monotone path of length ≤ 1. The weights are nonnegative, and satisfy $\sum\limits_{\boldsymbol{v}' \in \mathcal{N}(\boldsymbol{v})} \kappa_{\boldsymbol{v}'} w_{\boldsymbol{v},\boldsymbol{v}'} = 1$.

5 Flux Estimation

Given $U^n_{\boldsymbol{i}}$ and $S^n_{\boldsymbol{i}}$, we want to compute a second-order accurate estimate of the fluxes: $F^{n+\frac{1}{2}}_{\boldsymbol{f}} \approx F^d(\boldsymbol{x}_0 + (\boldsymbol{i} + \frac{1}{2}\boldsymbol{e}^d)h, t^n + \frac{1}{2}\Delta t)$. Specifically, we want to compute the fluxes at the center of the Cartesian grid faces corresponding to the faces of the embedded boundary geometry. In addition, we want to compute fluxes at the centers of Cartesian grid faces corresponding to faces adjacent to vofs, but that are completely covered. Pointwise operations are conceptually the same for both regular and irregular VoFs. In other operations we specify both the regular and irregular VoF calculation. The transformations $\nabla_U W$ and $\nabla_W U$ are functions of both space and time. We shall leave the precise centering of these transformations vague as this will be application-dependent. In outline, the method is given as follows.

5.1 Flux Estimation in Two Dimensions

1. Transform to primitive variables.

$$W^n_{\boldsymbol{v}} = W(U^n_{\boldsymbol{v}}) \tag{19}$$

2. Compute slopes $\Delta^d W_{\boldsymbol{v}}$. This is described separately in section 6.

3. Compute the effect of the normal derivative terms and the source term on the extrapolation in space and time from cell centers to faces. For $0 \leq d < \mathbf{D}$,

$$\begin{aligned} W_{\boldsymbol{v},\pm,d} &= W^n_{\boldsymbol{v}} + \frac{1}{2}(\pm I - \frac{\Delta t}{h} A^d_{\boldsymbol{v}}) P_\pm(\Delta^d W_{\boldsymbol{v}}) \\ A^d_{\boldsymbol{v}} &= A^d(W_{\boldsymbol{v}}) \\ P_\pm(W) &= \sum_{\pm\lambda_k > 0} (l_k \cdot W) r_k \\ W_{\boldsymbol{v},\pm,d} &= W_{\boldsymbol{v},\pm,d} + \frac{\Delta t}{2} \nabla_U W \cdot S^n_{\boldsymbol{v}} \end{aligned} \tag{20}$$

where λ_k are eigenvalues of $A^d_{\boldsymbol{i}}$, and l_k and r_k are the corresponding left and right eigenvectors. We then extrapolate to the covered faces. First we define the VoFs involved.

$$\begin{aligned} d' &= 1 - d \\ s^d &= sign(n^d) \\ \boldsymbol{v}^{up} &= ind^{-1}(ind(\boldsymbol{v}) + s^{d'}\boldsymbol{e}^{d'} - s^d \boldsymbol{e}^d) \\ \boldsymbol{v}^{side} &= ind^{-1}(ind(\boldsymbol{v}) + s^d \boldsymbol{e}^d) \\ \boldsymbol{v}^{corner} &= ind^{-1}(ind(\boldsymbol{v}) + s^{d'}\boldsymbol{e}^{d'}) \end{aligned} \tag{21}$$

Define $W^{\text{up,side,corner}}$, extrapolations to the edges near the VoFs near $\boldsymbol{v}$.

$$\begin{aligned} W^{\text{up}} &= W_{\boldsymbol{v}^{\text{up}},\mp,d} \\ W^{\text{side}} &= W_{\boldsymbol{v}^{\text{side}},\mp,d} - s^d \Delta^d W \\ W^{\text{corner}} &= W_{\boldsymbol{v}^{\text{corner}},\mp,d} \\ \Delta^d W &= \begin{cases} \Delta^d W^n_{\boldsymbol{v}^{\text{side}}} & \text{if } n^d > n^{d'} \\ \Delta^d W^n_{\boldsymbol{v}^{\text{corner}}} & \text{otherwise} \end{cases} \\ \Delta^{d'} W &= \begin{cases} \Delta^{d'} W^n_{\boldsymbol{v}^{\text{corner}}} & \text{if } n^d > n^{d'} \\ \Delta^{d'} W^n_{\boldsymbol{v}^{\text{up}}} & \text{otherwise} \end{cases} \end{aligned} \tag{22}$$

where the slopes are defined in section 6 If any of these vofs does not have a monotone path to the original VoF $\boldsymbol{v}$, we drop order the order of interpolation.

If $|n_d| < |n_{d'}|$:

$$W^{\text{full}} = \frac{|n_d|}{|n_{d'}|} W^{\text{corner}} + (1 - \frac{|n_d|}{|n_{d'}|}) W^{\text{up}} - (\frac{|n^d|}{|n^{d'}|} s^d \Delta^d W + s^{d'} \Delta^{d'} W) \tag{23}$$

$$W^{\text{covered}}_{\boldsymbol{v},\pm,d} = \begin{cases} W^{\text{full}} & \text{if both exist} \\ W^{\text{up}} & \text{if only } \boldsymbol{v}^{\text{up}} \text{ exists} \\ W^{\text{corner}} & \text{if only } \boldsymbol{v}^{\text{corner}} \text{ exists} \\ W^n_{\boldsymbol{v}} & \text{if neither exists} \end{cases} \tag{24}$$

If $|n_d| \geq |n_{d'}|$:

$$W^{\text{full}} = \frac{|n_{d'}|}{|n_d|} W^{\text{corner}} + (1 - \frac{|n_{d'}|}{|n_d|}) W^{\text{side}} - (\frac{|n^{d'}|}{|n^d|} s^{d'} \Delta^{d'} W + s^d \Delta^d W) \tag{25}$$

$$W^{\text{covered}}_{\boldsymbol{v},\pm,d} = \begin{cases} W^{\text{full}} & \text{if both exist} \\ W^{\text{side}} & \text{if only } \boldsymbol{v}^{\text{side}} \text{ exists} \\ W^{\text{corner}} & \text{if only } \boldsymbol{v}^{\text{corner}} \text{ exists} \\ W^n_{\boldsymbol{v}} & \text{if neither exists} \end{cases} \tag{26}$$

4. Compute estimates of F^d suitable for computing 1D flux derivatives $\frac{\partial F^d}{\partial x^d}$ using a Riemann solver for the interior, R, and for the boundary, R_B.

$$\begin{aligned} F^{\text{1D}}_{\boldsymbol{f}} &= R(W_{\boldsymbol{v}_-(\boldsymbol{f}),+,d}, W_{\boldsymbol{v}_+(\boldsymbol{f}),-,d}, d) \\ &\quad |\ \ R_B(W_{\boldsymbol{v}_-(\boldsymbol{f}),+,d}, (\boldsymbol{i} + \frac{1}{2}\boldsymbol{e}^d)h, d) \\ &\quad |\ \ R_B(W_{\boldsymbol{v}_+(\boldsymbol{f}),-,d}, (\boldsymbol{i} + \frac{1}{2}\boldsymbol{e}^d)h, d) \\ d &= dir(\boldsymbol{f}) \end{aligned} \tag{27}$$

5. Compute the covered fluxes $F^{\text{1D,covered}}$

$$\begin{aligned} F^{\text{1D, covered}}_{\boldsymbol{v},+,d} &= R(W_{\boldsymbol{v},+,d}, W^{\text{covered}}_{\boldsymbol{v},+,d}, d) \\ F^{\text{1D, covered}}_{\boldsymbol{v},-,d} &= R(W^{\text{covered}}_{\boldsymbol{v},-,d}, W_{\boldsymbol{v},-,d}, d) \end{aligned} \tag{28}$$

6. Compute final corrections to $W_{\boldsymbol{i},\pm,d}$ due to the final transverse derivatives. For regular cells, this takes the following form.

$$W^{n+\frac{1}{2}}_{\boldsymbol{i},\pm,d} = nW_{\boldsymbol{i},\pm,d} \ - \ \frac{\Delta t}{2h} \nabla_U W \cdot (F^{\text{1D}}_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^{d_1}} - F^{\text{1D}}_{\boldsymbol{i}-\frac{1}{2}\boldsymbol{e}^{d_1}}) \tag{29}$$

$$d \neq d_1, \ 0 \leq d, d_1 < \mathbf{D} \tag{30}$$

For irregular cells, we compute the transverse derivatives and use them to correct the extrapolated values of U and obtain time-centered fluxes at centers of Cartesian faces. In two dimensions, this takes the form

$$
\begin{aligned}
D^{d,\perp}F_{\boldsymbol{v}} &= \frac{1}{h}(\bar{F}_{\boldsymbol{v},+,d_1} - \bar{F}_{\boldsymbol{v},-,d_1}) \\
\bar{F}_{\boldsymbol{v},\pm,d'} &= \begin{cases} \frac{1}{N_{\boldsymbol{v},\pm,d'}} \sum_{\boldsymbol{f}\in\mathcal{F}_{\boldsymbol{v},\pm,d'}} F^{1D}_{\boldsymbol{f},\pm,d'} & \text{if } N_{\boldsymbol{v},\pm,d'} > 0 \\ F^{\text{1D, covered}}_{\boldsymbol{v},\pm,d'} & \text{otherwise} \end{cases} \\
d \neq d_1, &\quad 0 \leq d, d_1 < \mathbf{D} \\
W^{n+\frac{1}{2}}_{\boldsymbol{v},\pm,d} &= W_{\boldsymbol{v},\pm,d} - \frac{\Delta t}{2}\nabla_U W(D^{d,\perp}F_{\boldsymbol{v}})
\end{aligned}
\tag{31}
$$

Extrapolate to covered faces with the procedure described above using $W^{n+\frac{1}{2}}_{\cdot,\mp,d}$ to form $W^{n+\frac{1}{2},\text{covered}}_{\cdot,\pm,d}$.

7. Compute the flux estimate.

$$
\begin{aligned}
F^{n+\frac{1}{2}}_{\boldsymbol{f}} &= R(W^{n+\frac{1}{2}}_{\boldsymbol{v}^-(\boldsymbol{f}),+,d}, W^{n+\frac{1}{2}}_{\boldsymbol{v}^+(\boldsymbol{f}),-,d}, d) \\
&\quad |\ \ R_B(W^{n+\frac{1}{2}}_{\boldsymbol{v}^-(\boldsymbol{f}),+,d}, (\boldsymbol{i} + \frac{1}{2}\boldsymbol{e}^d)h, d) \\
&\quad |\ \ R_B(W^{n+\frac{1}{2}}_{\boldsymbol{v}^+(\boldsymbol{f}),-,d}, (\boldsymbol{i} + \frac{1}{2}\boldsymbol{e}^d)h, d) \\
F^{n+\frac{1}{2},\text{covered}}_{\boldsymbol{v},-,d} &= R(W^{n+\frac{1}{2},\text{covered}}_{\boldsymbol{v},+,d}, W^{n+\frac{1}{2}}_{\boldsymbol{v},-,d}, d) \\
F^{n+\frac{1}{2},\text{covered}}_{\boldsymbol{v},+,d} &= R(W^{n+\frac{1}{2}}_{\boldsymbol{v},+,d}, W^{n+\frac{1}{2},\text{covered}}_{\boldsymbol{v},+,d}, d)
\end{aligned}
\tag{32}
$$

8. Modify the flux with artificial viscosity where the flow is compressive.

5.2 Flux Estimation in Three Dimensions

1. Transform to primitive variables.

$$
W^n_{\boldsymbol{v}} = W(U^n_{\boldsymbol{v}}) \tag{33}
$$

2. Compute slopes $\Delta^d W_{\boldsymbol{v}}$. This is described separately in section 6.

3. Compute the effect of the normal derivative terms and the source term on the

extrapolation in space and time from cell centers to faces. For $0 \leq d < \mathbf{D}$,

$$
\begin{aligned}
W_{\boldsymbol{v},\pm,d} &= W^n_{\boldsymbol{v}} + \frac{1}{2}(\pm I - \frac{\Delta t}{h}A^d_{\boldsymbol{v}})P_\pm(\Delta^d W_{\boldsymbol{v}}) \\
A^d_{\boldsymbol{v}} &= A^d(W_{\boldsymbol{v}}) \\
P_\pm(W) &= \sum_{\pm\lambda_k > 0}(l_k \cdot W)r_k \\
W_{\boldsymbol{v},\pm,d} &= W_{\boldsymbol{v},\pm,d} + \frac{\Delta t}{2}\nabla_U W \cdot S^n_{\boldsymbol{v}}
\end{aligned}
\tag{34}
$$

where λ_k are eigenvalues of $A^d_{\boldsymbol{i}}$, and l_k and r_k are the corresponding left and right eigenvectors.

We then extrapolate to the covered faces. Define the direction of the face normal to be d_f and d_1, d_2 to be the directions tangential to the face. The procedure develops as follows

- We define the associated vofs.
- We form a 2x2 grid of values along a plane h away from the covered face and bilinearly interpolate to the point where the normal intersects the plane.
- We use the slopes of the solution to extrapolate along the normal to get a second-order approximation of the solution at the covered face.

Which plane is selected is determined by the direction of the normal. If any of these VoFs does not have a monotone path to the original VoF $\boldsymbol{v}$, we drop order the order of interpolation.

If $|n_f| \geq |n_{d_1}|$ and $|n_{d_f}| \geq |n_{d_2}|$:

$$
\begin{aligned}
\boldsymbol{v}^{00} &= ind^{-1}(ind(\boldsymbol{v}) + s^{d_f}\boldsymbol{e}^{d_f}) \\
\boldsymbol{v}^{10} &= ind^{-1}(ind(\boldsymbol{v}) + s^{d_1}\boldsymbol{e}^{d_1}) \\
\boldsymbol{v}^{01} &= ind^{-1}(ind(\boldsymbol{v}) + s^{d_2}\boldsymbol{e}^{d_2}) \\
\boldsymbol{v}^{11} &= ind^{-1}(ind(\boldsymbol{v}) + s^{d_1}\boldsymbol{e}^{d_1} + s^{d_2}\boldsymbol{e}^{d_2}) \\
W^{00} &= W_{\boldsymbol{v}^{00},\mp,d_f} - s^{d_f}\Delta^{d_f}W_{\boldsymbol{v}^{00}} \\
W^{10} &= W_{\boldsymbol{v}^{10},\mp,d_f} \\
W^{01} &= W_{\boldsymbol{v}^{01},\mp,d_f} \\
W^{11} &= W_{\boldsymbol{v}^{11},\mp,d_f}
\end{aligned}
\tag{35}
$$

We form a bilinear function $W(x_{d_1}, x_{d_2})$ in the plane formed by the four faces at which the values live:

$$
\begin{aligned}
W(x_{d_1}, x_{d_2}) = Ax_{d_1} + Bx_{d_2} + Cx_{d_1}x_{d_2} + D \\
A =& s^{d_1}(W^{10} - W^{00}) \\
B =& s^{d_2}(W^{01} - W^{00}) \\
C =& s^{d_1}s^{d_2}(W^{11} - W^{00}) - (W^{10} - W^{00}) - (W^{01} - W^{00}) \\
D =& W^{00}
\end{aligned}
\tag{36}
$$

We then extrapolate to the covered face from the point on the plane where the normal intersects

$$
W^{\mathsf{full}} = W(s^{d_1}\frac{|n_{d_1}|}{|n_{d_f}|}, s^{d_2}\frac{|n_{d_2}|}{|n_{d_f}|}) - \Delta^{d_f}W_{\boldsymbol{v}^{00}} - s^{d_1}\frac{|n_{d_1}|}{|n_{d_f}|}\Delta^{d_1}W_{\boldsymbol{v}^{10}} - s^{d_2}\frac{|n_{d_2}|}{|n_{d_f}|}\Delta^{d_2}W_{\boldsymbol{v}^{01}}
\tag{37}
$$

Otherwise (assume $|n_{d_1}| \geq |n_{d_f}|$ and $|n_{d_1}| \geq |n_{d_2}|$):

$$
\begin{aligned}
\boldsymbol{v}^{00} =& ind^{-1}(ind(\boldsymbol{v}) + s^{d_1}\boldsymbol{e}^{d_1}) \\
\boldsymbol{v}^{10} =& ind^{-1}(ind(\boldsymbol{v}) + s^{d_1}\boldsymbol{e}^{d_1}) - s^{d_f}\boldsymbol{e}^{d_f} \\
\boldsymbol{v}^{01} =& ind^{-1}(ind(\boldsymbol{v}) + s^{d_1}\boldsymbol{e}^{d_1}) + s^{d_2}\boldsymbol{e}^{d_2} \\
\boldsymbol{v}^{11} =& ind^{-1}(ind(\boldsymbol{v}) + s^{d_1}\boldsymbol{e}^{d_1} - s^{d_f}\boldsymbol{e}^{d_f} + s^{d_2}\boldsymbol{e}^{d_2} \\
W^{00} &= W_{\boldsymbol{v}^{00},\mp,d_f} \\
W^{10} &= W_{\boldsymbol{v}^{10},\mp,d_f} \\
W^{01} &= W_{\boldsymbol{v}^{01},\mp,d_f} \\
W^{11} &= W_{\boldsymbol{v}^{11},\mp,d_f}
\end{aligned}
\tag{38}
$$

We form a bilinear function $W(x_{d_1}, x_{d_2})$ in the plane formed by the four faces at which the values live. This is shown in equation 36. We then extrapolate to the covered face from the point on the plane where the normal intersects

$$
W^{\mathsf{full}} = W(s^{d_f}\frac{|n_{d_f}|}{|n_{d_1}|}, s^{d_2}\frac{|n_{d_2}|}{|n_{d_1}|}) - \Delta^{d_1}W_{\boldsymbol{v}^{00}} - s^{d_f}\frac{|n_{d_f}|}{|n_{d_1}|}\Delta^{d_f}W_{\boldsymbol{v}^{10}} - s^{d_2}\frac{|n_{d_2}|}{|n_{d_1}|}\Delta^{d_2}W_{\boldsymbol{v}^{01}}
\tag{39}
$$

In either case,

$$
W^{\mathsf{covered}}_{\boldsymbol{v},\pm,d} = \begin{cases} W^{\mathsf{full}} & \text{if all four VoFs exist} \\ W^n_{\boldsymbol{v}} & \text{otherwise} \end{cases}
\tag{40}
$$

4. Compute estimates of F^d suitable for computing 1D flux derivatives $\frac{\partial F^d}{\partial x^d}$ using a

Riemann solver for the interior, R, and for the boundary, R_B.

$$\begin{aligned} F^{\text{1D}}_{\boldsymbol{f}} &= R(W_{\boldsymbol{v}-(\boldsymbol{f}),+,d}, W_{\boldsymbol{v}+(\boldsymbol{f}),-,d}, d) \\ &\quad \mid\ R_B(W_{\boldsymbol{v}-(\boldsymbol{f}),+,d}, (\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d)h, d) \\ &\quad \mid\ R_B(W_{\boldsymbol{v}+(\boldsymbol{f}),-,d}, (\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d)h, d) \\ d &= dir(\boldsymbol{f}) \end{aligned} \tag{41}$$

5. Compute the covered fluxes $F^{\text{1D,covered}}$

$$\begin{aligned} F^{\text{1D, covered}}_{\boldsymbol{v},+,d} &= R(W_{\boldsymbol{v},+,d}, W^{\text{covered}}_{\boldsymbol{v},+,d}, d) \\ F^{\text{1D, covered}}_{\boldsymbol{v},-,d} &= R(W^{\text{covered}}_{\boldsymbol{v},-,d}, W_{\boldsymbol{v},-,d}, d) \end{aligned} \tag{42}$$

6. Compute corrections to $U_{\boldsymbol{i},\pm,d}$ corresponding to one set of transverse derivatives appropriate to obtain $(1,1,1)$ diagonal coupling. This step is only meaningful in three dimensions. We compute 1D flux differences, and use them to compute $U_{\boldsymbol{v},\pm,d_1,d_2}$, the d_1-edge-centered state partially updated by the effect of derivatives in the d_1, d_2 directions.

$$\begin{aligned} D^{\text{1D}}_d F^{\text{1D}}_{\boldsymbol{v}} &= \frac{1}{h}(\bar{F}^{\text{1D}}_{\boldsymbol{v},+,d} - \bar{F}^{\text{1D}}_{\boldsymbol{v},-,d}) \\ \bar{F}_{\boldsymbol{v},\pm,d} &= \begin{cases} \frac{1}{N_{\pm,d}}(\sum\limits_{\boldsymbol{f}\in\mathcal{F}_{\boldsymbol{v},\pm,d}} F^{\text{1D}}_{\boldsymbol{f}}) & \text{if } N_{\boldsymbol{v},\pm,d} > 0 \\ F^{\text{1D, covered}}_{\boldsymbol{v},\pm,d} & \text{otherwise} \end{cases} \end{aligned} \tag{43}$$

$$W_{\boldsymbol{v},\pm,d_1,d_2} = W_{\boldsymbol{v},\pm,d_1} - \frac{\Delta t}{3}\nabla_U W(D^{\text{1D}}_{d_2}F^{\text{1D}})_{\boldsymbol{v}} \tag{44}$$

We then extrapolate to covered faces with the procedure described above using $W_{\cdot,\pm,d_1,d_2}$ to form $W^{\text{covered},d}_{\cdot,\pm,d_1,d_2}$ and compute an estimate to the fluxes:

$$\begin{aligned} F_{\boldsymbol{f},d_1,d_2} &= R(W_{\boldsymbol{v}-(\boldsymbol{f}),+,d_1,d_2}, W_{\boldsymbol{v}+(\boldsymbol{f}),-,d_1,d_2}, d_1) \\ &\quad \mid\ R_B(W_{\boldsymbol{v}-(\boldsymbol{f}),+,d_1,d_2}, (\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d)h, d_1) \\ &\quad \mid\ R_B(W_{\boldsymbol{v}+(\boldsymbol{f}),-,d_1,d_2}, (\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d)h, d_1) \\ d &= dir(\boldsymbol{f}) \\ F^{\text{covered}}_{\boldsymbol{v},-,d_1,d_2} &= R(W^{\text{covered}}_{\boldsymbol{v},-,d1,d2}, W_{\boldsymbol{v},-,d_1,d_2}, d_1) \\ F^{\text{covered}}_{\boldsymbol{v},+,d_1,d_2} &= R(W_{\boldsymbol{v},+,d1,d2}, W^{\text{covered}}_{\boldsymbol{v},+,d_1,d_2}, d_1) \end{aligned} \tag{45}$$

7. Compute final corrections to $W_{\boldsymbol{i},\pm,d}$ due to the final transverse derivatives. We compute the $2\mathbf{D}$ transverse derivatives and use them to correct the extrapolated values of U and obtain time-centered fluxes at centers of Cartesian faces. In three dimensions, this takes the form:

$$\begin{aligned}
D^{d,\perp}F_{\boldsymbol{v}} &= \frac{1}{h}(\bar{F}_{\boldsymbol{v},+,d_1,d_2} - \bar{F}_{\boldsymbol{v},-,d_1,d_2} + \bar{F}_{\boldsymbol{v},+,d_2,d_1} - \bar{F}_{\boldsymbol{v},-,d_2,d_1}) \\
\bar{F}_{\boldsymbol{v},\pm,d',d''} &= \begin{cases} \frac{1}{N_{\boldsymbol{v},\pm,d'}} \sum_{\boldsymbol{f}\in\mathcal{F}_{\boldsymbol{v},\pm,d'}} F_{\boldsymbol{f},\pm,d',d''} & \text{if } N_{\boldsymbol{v},\pm,d'} > 0 \\ F^{\text{covered}}_{\boldsymbol{v},\pm,d',d''} & \text{otherwise} \end{cases} \\
d \neq d_1 \neq d_2 &\quad 0 \leq d, d_1, d_2 < \mathbf{D} \\
W^{n+\frac{1}{2}}_{\boldsymbol{v},\pm,d} &= W_{\boldsymbol{v},\pm,d} - \frac{\Delta t}{2}\nabla_U W(D^{d,\perp}F_{\boldsymbol{v}})
\end{aligned} \tag{46}$$

   We then extrapolate to covered faces with the procedure described above using $W^{n+\frac{1}{2}}_{\cdot,\pm,d}$ to form $W^{n+\frac{1}{2},\text{covered},d}_{\cdot,\pm,d}$.

8. Compute the flux estimate.

$$\begin{aligned}
F^{n+\frac{1}{2}}_{\boldsymbol{f}} &= R(W^{n+\frac{1}{2}}_{\boldsymbol{v}^-(\boldsymbol{f}),+,d}, W^{n+\frac{1}{2}}_{\boldsymbol{v}^+(\boldsymbol{f}),-,d}, d) \\
&\quad | \ \ R_B(W^{n+\frac{1}{2}}_{\boldsymbol{v}^-(\boldsymbol{f}),+,d}, (\boldsymbol{i} + \frac{1}{2}\boldsymbol{e}^d)h, d) \\
&\quad | \ \ R_B(W^{n+\frac{1}{2}}_{\boldsymbol{v}^+(\boldsymbol{f}),-,d}, (\boldsymbol{i} + \frac{1}{2}\boldsymbol{e}^d)h, d) \\
F^{n+\frac{1}{2},\text{covered}}_{\boldsymbol{v},-,d} &= R(W^{n+\frac{1}{2},\text{covered}}_{\boldsymbol{v},+,d}, W^{n+\frac{1}{2}}_{\boldsymbol{v},-,d}, d) \\
F^{n+\frac{1}{2},\text{covered}}_{\boldsymbol{v},+,d} &= R(W^{n+\frac{1}{2}}_{\boldsymbol{v},+,d}, W^{n+\frac{1}{2},\text{covered}}_{\boldsymbol{v},+,d}, d)
\end{aligned} \tag{47}$$

9. Modify the flux with artificial viscosity where the flow is compressive.

5.3 Modificiations for R-Z Computations

For R-Z calculations, we make some adjustments to the algorithm. Specifically, we separate the radial pressure force as a separate flux. This makes free-stream preservation in the radial direction easier to achieve. For this section, we will confine ourselves to the compressible Euler equations.

5.3.1 Equations of Motion

The compressible Euler equations in R-Z coordinates are given by

$$\frac{\partial U}{\partial t} + \frac{1}{r}\frac{\partial (rF^r)}{\partial r} + \frac{1}{r}\frac{\partial (rF^z)}{\partial z} + \frac{\partial H}{\partial r} + \frac{\partial H}{\partial z} = 0 \tag{48}$$

where

$$
\begin{aligned}
U =& (\rho, \rho u_r, \rho u_z, \rho E)^T \\
F^r =& (\rho u_r, \rho u_r^2, \rho u_r u_z, \rho u_r(E+p))^T \\
F^z =& (\rho u_z, \rho u_r u_z, \rho u_z^2, \rho u_z(E+p))^T \\
H =& (0, p, p, 0)^T
\end{aligned}
\tag{49}
$$

5.3.2 Flux Divergence Approximations

In section 4, we describe our solution update strategy and this remains largely unchanged. Our update still takes the form of equation 16 and redistribution still takes the form of equation 18. The definitions of the divergence approximations do change, however. The volume of a full cell ΔV_j is given by

$$
\Delta V_j = (j + \frac{1}{2})h^3 \tag{50}
$$

where $(i,j) = ind^{-1}(\boldsymbol{v})$. Define $\kappa_{\boldsymbol{v}}^{vol}$ to be the real volume of the cell that the VoF occupies.

$$
\kappa_{\boldsymbol{v}}^{vol} = \frac{1}{\Delta V}\int_{\Delta_{\boldsymbol{v}}} r dr dz = \frac{1}{\Delta V}\int_{\partial\Delta_{\boldsymbol{v}}} \frac{r^2}{2} n_r dl \tag{51}
$$

$$
\kappa_{\boldsymbol{v}}^{vol} = \frac{h}{2\Delta V}((\alpha r^2)_{\boldsymbol{f}(\boldsymbol{v},+,r)} - (\alpha r^2)_{\boldsymbol{f}(\boldsymbol{v},-,r)} - \alpha_B \bar{r}_{\delta\boldsymbol{v}}^2 n^r) \tag{52}
$$

The conservative divergence of the flux in RZ is given by

$$
\begin{aligned}
(D\cdot\vec{F})_{\boldsymbol{v}}^c = \frac{h}{\Delta V \kappa_{\boldsymbol{v}}^{vol}}&((r\bar{F}^r\alpha)_{\boldsymbol{f}(\boldsymbol{v},+,r)} - (r\bar{F}^r\alpha)_{\boldsymbol{f}(\boldsymbol{v},-,r)} \\
&+(\bar{r}\bar{F}^z\alpha)_{\boldsymbol{f}(\boldsymbol{v},+,z)} - (\bar{r}\bar{F}^z\alpha)_{\boldsymbol{f}(\boldsymbol{v},-,z)})
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial H}{\partial r}\right)^c &= \frac{1}{\kappa_{\boldsymbol{v}} h^2}\int \frac{\partial H}{\partial r} dr dz = \frac{1}{\kappa_{\boldsymbol{v}} h^2}\int H n_r dl \\
\left(\frac{\partial H}{\partial z}\right)^c &= \frac{1}{\kappa_{\boldsymbol{v}} h^2}\int \frac{\partial H}{\partial z} dr dz = \frac{1}{\kappa_{\boldsymbol{v}} h^2}\int H n_z dl
\end{aligned}
$$

We always deal with these divergences in a form multiplied by the volume fraction κ.

$$
\begin{aligned}
\kappa_{\boldsymbol{v}}(D\cdot\vec{F})_{\boldsymbol{v}}^c = \frac{h\kappa_{\boldsymbol{v}}}{\Delta V \kappa_{\boldsymbol{v}}^{vol}}&((r\bar{F}^r\alpha)_{\boldsymbol{f}(\boldsymbol{v},+,r)} - (r\bar{F}^r\alpha)_{\boldsymbol{f}(\boldsymbol{v},-,r)} \\
&+(\bar{r}\bar{F}^z\alpha)_{\boldsymbol{f}(\boldsymbol{v},+,z)} - (\bar{r}\bar{F}^z\alpha)_{\boldsymbol{f}(\boldsymbol{v},-,z)})
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{\boldsymbol{v}}\left(\frac{\partial H}{\partial r}\right)^c =& \frac{1}{h^2}\int H n_r dl = \frac{1}{h}((H\alpha)_{\boldsymbol{f}(\boldsymbol{v},+,r)} - (H\alpha)_{\boldsymbol{f}(\boldsymbol{v},-,r)}) \\
\kappa_{\boldsymbol{v}}\left(\frac{\partial H}{\partial z}\right)^c =& \frac{1}{h^2}\int H n_z dl = \frac{1}{h}((H\alpha)_{\boldsymbol{f}(\boldsymbol{v},+,z)} - (H\alpha)_{\boldsymbol{f}(\boldsymbol{v},-,z)})
\end{aligned}
$$

where $\bar{F}$ has been interpolated to face centroids where α denotes the ordinary area fraction. The nonconservative divergence of the flux in RZ is given by

$$\begin{aligned}(D \cdot \vec{F})^{nc}_{\boldsymbol{v}} = & \frac{1}{hr_{\boldsymbol{v}}}((rF^r)_{\boldsymbol{f}(\boldsymbol{v},+,r)} - (rF^r)_{\boldsymbol{f}(\boldsymbol{v},-,r)}) \\ & + \frac{1}{h}(F^z_{\boldsymbol{f}(\boldsymbol{v},+,z)} - F^z_{\boldsymbol{f}(\boldsymbol{v},-,z)})\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial H}{\partial r}\right)^{nc} &= \frac{1}{h}(H_{\boldsymbol{f}(\boldsymbol{v},+,r)} - H_{\boldsymbol{f}(\boldsymbol{v},-,r)}) \\ \left(\frac{\partial H}{\partial z}\right)^{nc} &= \frac{1}{h}(H_{\boldsymbol{f}(\boldsymbol{v},+,z)} - H_{\boldsymbol{f}(\boldsymbol{v},-,z)})\end{aligned}$$

5.3.3 Primitive Variable Form of the Equations

In the predictor step, we use the nonconservative form of the equations of motion. See Courant and Friedrichs [CF48] for derivations.

$$\frac{\partial W}{\partial t} + A^r \frac{\partial W}{\partial r} + A^z \frac{\partial W}{\partial z} = S \tag{53}$$

where

$$\begin{aligned}W &= (\rho, u_r, u_z, p)^T \\ S &= \left(-\rho \frac{u_r}{r}, 0, 0, -\rho c^2 \frac{u_r}{r}\right)^T \\ A^r &= \begin{pmatrix} u_r & \rho & 0 & 0 \\ 0 & u_r & 0 & \frac{1}{\rho} \\ 0 & 0 & u_r & 0 \\ 0 & \rho c^2 & 0 & u_r \end{pmatrix} \\ A^r &= \begin{pmatrix} u_z & \rho & 0 & 0 \\ 0 & u_z & 0 & 0 \\ 0 & 0 & u_z & \frac{1}{\rho} \\ 0 & 0 & \rho c^2 & u_z \end{pmatrix}\end{aligned}$$

5.3.4 Flux Registers

Refluxing is the balancing the fluxes at coarse-fine interfaces so the coarse side of the interface is using the same flux as the integral of the fine fluxes over the same area. In this way, we maintain strong mass conservation at coarse-fine interfaces. As shown in equation, 5.3.2, the conservative divergence in cylindrical coordinates is has a differenct form than in Cartesian coordinates. It is therefore necessary to describe the refluxing operation specifically for cylindrical coordinates.

Let $\vec{F}^{comp} = \{\vec{F}^f, \vec{F}^{c,valid}\}$ be a two-level composite vector field. We want to define a composite divergence $D^{comp}(\vec{F}^f, \vec{F}^{c,valid})_{\boldsymbol{v}}$, for $\boldsymbol{v} \in V^c_{valid}$. We do this by extending $F^{c,valid}$ to the faces adjacent to $\boldsymbol{v} \in V^c_{valid}$, but are covered by $\mathcal{F}^f_{valid}$.

$$
\begin{aligned}
< F^f_z >_{\boldsymbol{f}_c} &= \left(\frac{\kappa_{\boldsymbol{v}_c}}{\kappa^{vol}_{\boldsymbol{v}_c}\Delta V_{\boldsymbol{v}_c}}\right)\left(\frac{h^2}{(n_{ref})^{(\mathbf{D}-1)}}\right)\sum_{\boldsymbol{f}\in\mathcal{C}^{-1}_{n_{ref}(\boldsymbol{f}_c)}}(\bar{r}\alpha)_{\boldsymbol{f}}(\bar{F}^z+\bar{H})_{\boldsymbol{f}} \\
< F^f_r >_{\boldsymbol{f}_c} &= \left(\frac{\kappa_{\boldsymbol{v}_c}}{\kappa^{vol}_{\boldsymbol{v}_c}\Delta V_{\boldsymbol{v}_c}}\right)\left(\frac{h^2}{(n_{ref})^{(\mathbf{D}-1)}}\right)\sum_{\boldsymbol{f}\in\mathcal{C}^{-1}_{n_{ref}}(\boldsymbol{f}_c)}(r\alpha)_{\boldsymbol{f}}(F^r+H)_{\boldsymbol{f}} \\
F^c_{r,\boldsymbol{f}_c} &= \left(\frac{\kappa_{\boldsymbol{v}_c}}{\kappa^{vol}_{\boldsymbol{v}_c}\Delta V_{\boldsymbol{v}_c}}\right)(h^2(r\alpha)_{\boldsymbol{f}_c})(F^r+H)_{\boldsymbol{f}_c} \\
F^c_{z,\boldsymbol{f}_c} &= \left(\frac{\kappa_{\boldsymbol{v}_c}}{\kappa^{vol}_{\boldsymbol{v}_c}\Delta V_{\boldsymbol{v}_c}}\right)(h^2(\bar{r}\alpha)_{\boldsymbol{f}_c})(\bar{F}^z+\bar{H})_{\boldsymbol{f}_c} \\
\boldsymbol{f}_c &\in ind^{-1}(\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d), \boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d \in \zeta^f_{d,+}\cup\zeta^f_{d,-} \\
\zeta^f_{d,\pm} &= \{\boldsymbol{i}\pm\frac{1}{2}\boldsymbol{e}^d : \boldsymbol{i}\pm\boldsymbol{e}^d \in \Omega^c_{valid}, \boldsymbol{i}\in\mathcal{C}_{n_{ref}}(\Omega^f)\}
\end{aligned}
$$

The VoF $\boldsymbol{v}_c$ is the coarse volume that is adjacent to the coarse-fine interface and $r_{\boldsymbol{v}_c}$ is the radius of its cell center. Then we can define $(D\cdot\vec{F})_{\boldsymbol{v}}, \boldsymbol{v}\in\mathcal{V}^c_{valid}$, using the expression above, with $\tilde{F}_{\boldsymbol{f}} =< F^f_d >$ on faces covered by $\mathcal{F}^f$. We can express the composite divergence in terms of a level divergence, plus a correction. We define a flux register $\delta\vec{F}^f$, associated with the fine level

$$
\begin{aligned}
\delta\vec{F}^f &= (\delta F^f_{0,\dots}\delta F^f_{D-1}) \\
\delta F^f_d &: ind^{-1}(\zeta^f_{d,+}\cup\zeta^f_{d,-}) \rightarrow \mathbb{R}^m
\end{aligned}
$$

If $\vec{F}^c$ is any coarse level vector field that extends $\vec{F}^{c,valid}$, i.e. $F^c_d = F^{c,valid}_d$ on $\mathcal{F}^{c,d}_{valid}$ then for $\boldsymbol{v}\in\mathcal{V}^c_{valid}$

$$D^{comp}(\vec{F}^f, \vec{F}^{c,valid})_{\boldsymbol{v}} = (D\vec{F}^c)_{\boldsymbol{v}} + D_R(\delta\vec{F}^c)_{\boldsymbol{v}} \tag{54}$$

Here $\delta\vec{F}^f$ is a flux register, set to be

$$\delta F^f_d =< F^f_d > -F^c_d \text{ on } ind^{-1}(\zeta^c_{d,+}\cup\zeta^c_{d,-}) \tag{55}$$

D_R is the reflux divergence operator. For valid coarse vofs adjacent to Ω^f it is given by

$$\kappa_{\boldsymbol{v}}(D_R\delta\vec{F}^f)_{\boldsymbol{v}} = \sum_{d=0}^{D-1}(\sum_{\boldsymbol{f}:\boldsymbol{v}=\boldsymbol{v}^+(\boldsymbol{f})}\delta F^f_{d,\boldsymbol{f}} - \sum_{\boldsymbol{f}:\boldsymbol{v}=\boldsymbol{v}^-(\boldsymbol{f})}\delta F^f_{d,\boldsymbol{f}}) \tag{56}$$

For the remaining vofs in $\mathcal{V}^f_{valid}$,

$$(D_R\delta\vec{F}^f) \equiv 0 \tag{57}$$

We then add the reflux divergence to adjust the coarse solution U^c to preserve conservation.

$$U^c_{\boldsymbol{v}} \mathrel{+}= \kappa_{\boldsymbol{v}}(D_R(\delta F))_{\boldsymbol{v}} \tag{58}$$

5.4 Artifical Viscosity

The artificial viscosity coefficient is K_0, the velocity is $\vec{u}$ and $d = dir(\boldsymbol{f})$.

$$\begin{aligned}
(D\vec{u})_{\boldsymbol{f}} &= (u^d_{\boldsymbol{v}^+(\boldsymbol{f})} - u^d_{\boldsymbol{v}^-(\boldsymbol{f})}) + \sum_{d' \neq d} \frac{1}{2}(\Delta^{d'} u^{d'}_{\boldsymbol{v}^+(\boldsymbol{f})} + \Delta^{d'} u^{d'}_{\boldsymbol{v}^-(\boldsymbol{f})}) \\
K_{\boldsymbol{f}} &= K_0 \ \mathsf{max}(-(D\vec{u})_{\boldsymbol{f}}, 0) \\
F^{n+\frac{1}{2}}_{\boldsymbol{f}} &= F^{n+\frac{1}{2}}_{\boldsymbol{f}} - K_{\boldsymbol{f}}(U^n_{\boldsymbol{v}^+(\boldsymbol{f})} - U^n_{\boldsymbol{v}^-(\boldsymbol{f})}) \\
F^{\mathsf{covered}}_{\boldsymbol{v},\pm,d} &= F^{\mathsf{covered}}_{\boldsymbol{v},\pm,d} - K_{\boldsymbol{f}}(U^n_{\boldsymbol{v}^+(\boldsymbol{f})} - U^n_{\boldsymbol{v}^-(\boldsymbol{f})})
\end{aligned}$$

We modify the covered face with the same divergence used in the adjacent uncovered face.

$$\begin{aligned}
F^{\mathsf{covered}}_{\boldsymbol{v},\pm,d} =& F^{\mathsf{covered}}_{\boldsymbol{v},\pm,d} - K_{\boldsymbol{f}}(U^n_{\boldsymbol{v}^+(\boldsymbol{f})} - U^n_{\boldsymbol{v}^-(\boldsymbol{f})}) \\
\boldsymbol{f} =& \boldsymbol{f}(\boldsymbol{v}, \mp, d)
\end{aligned}$$

This has the effect of negating the effect of artificial viscosity on the non-conservative divergence of the flux at irregular cells. We describe later that the solid wall boundary condition at the embedded boundary is also modified with artificial viscosity.

6 Slope Calculation

We will use the 4th order slope calculation in Colella and Glaz [CG85] combined with characteristic limiting.

$$
\begin{aligned}
\Delta^d W_{\boldsymbol{v}} &= \zeta_{\boldsymbol{v}} \; \widetilde{\Delta}^d W_{\boldsymbol{v}} \\
\widetilde{\Delta}^d W_{\boldsymbol{v}} &= \Delta^{vL}(\Delta^B W_{\boldsymbol{v}}, \Delta^L W_{\boldsymbol{v}}, \Delta^R W_{\boldsymbol{v}}) \mid \Delta_2^d W_{\boldsymbol{v}} \mid \Delta_2^d W_{\boldsymbol{v}} \\
\Delta_2^d W_{\boldsymbol{v}} &= \Delta^{vL}(\Delta^C W_{\boldsymbol{v}}, \Delta^L W_{\boldsymbol{v}}, \Delta^R W_{\boldsymbol{v}}) \mid \Delta^{VLL} W_{\boldsymbol{v}} \mid \Delta^{VLR} W_{\boldsymbol{v}} \\
\Delta^B W_{\boldsymbol{v}} &= \frac{2}{3}((W - \frac{1}{4}\Delta_2^d W) \ll \boldsymbol{e}^d)_{\boldsymbol{v}} - ((W + \frac{1}{4}\Delta_2^d W) \ll -\boldsymbol{e}^d)_{\boldsymbol{v}}) \\
\Delta^C W_{\boldsymbol{v}} &= \frac{1}{2}((W^n \ll \boldsymbol{e}^d)_{\boldsymbol{v}} - (W^n \ll -\boldsymbol{e}^d)_{\boldsymbol{v}}) \\
\Delta^L W_{\boldsymbol{v}} &= W^n_{\boldsymbol{v}} - (W^n \ll -\boldsymbol{e}^d)_{\boldsymbol{v}} \\
\Delta^R W_{\boldsymbol{v}} &= (W^n \ll \boldsymbol{e}^d)_{\boldsymbol{v}} - W^n_{\boldsymbol{v}} \\
\Delta^{3L} W_{\boldsymbol{v}} &= \frac{1}{2}(3W^n_{\boldsymbol{v}} - 4(W^n \ll -\boldsymbol{e}^d)_{\boldsymbol{v}} + (W^n \ll -2\boldsymbol{e}^d)_{\boldsymbol{v}}) \\
\Delta^{3R} W_{\boldsymbol{v}} &= \frac{1}{2}(-3W^n_{\boldsymbol{v}} + 4(W^n \ll \boldsymbol{e}^d)_{\boldsymbol{v}} - (W^n \ll 2\boldsymbol{e}^d)_{\boldsymbol{v}}) \\
\Delta^{VLL} W_{\boldsymbol{v}} &= \begin{cases} \text{min}(\Delta^{3L} W_{\boldsymbol{v}}, \Delta^L_{\boldsymbol{v}}) & \text{if } \Delta^{3L} W_{\boldsymbol{v}} \cdot \Delta^L W_{\boldsymbol{v}} > 0 \\ 0 & \text{otherwise} \end{cases} \\
\Delta^{VLR} W_{\boldsymbol{v}} &= \begin{cases} \text{min}(\Delta^{3R} W_{\boldsymbol{v}}, \Delta^R_{\boldsymbol{v}}) & \text{if } \Delta^{3R} W_{\boldsymbol{v}} \cdot \Delta^R W_{\boldsymbol{v}} > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

At domain boundaries, $\Delta^L W_{\boldsymbol{v}}$ and $\Delta^R W_{\boldsymbol{v}}$ may be overwritten by the application. There are two versions of the van Leer limiter $\Delta^{vL}(\delta W_C, \delta W_L, \delta W_R)$ that are commonly used. One is to apply a limiter to the differences in characteristic variables.

1. Compute expansion of one-sided and centered differences in characteristic variables.

$$\alpha^k_L = l^k \cdot \delta W_L \tag{59}$$

$$\alpha^k_R = l^k \cdot \delta W_R \tag{60}$$

$$\alpha^k_C = l^k \cdot \delta W \tag{61}$$

2. Apply van Leer limiter

$$\alpha^k = \begin{cases} \text{min}(2 \,|\, \alpha^k_L \,|,\; 2 \,|\, \alpha^k_R \,|,\; |\, \alpha^k_C \,|) & \text{if } \alpha^k_L \cdot \alpha^k_R > 0 \\ 0 & \text{otherwise} \end{cases} \tag{62}$$

3. $\Delta^{vL} = \sum_k \alpha^k r^k$

Here, $l^k = l^k(W_{\boldsymbol{i}}^n)$ and $r^k = r^k(W_{\boldsymbol{i}}^n)$.

For a variety of problems, it suffices to apply the van Leer limiter componentwise to the differences. Formally, this can be obtain from the more general case above by taking the matrices of left and right eigenvectors to be the identity.

Finally, we give the algorithm for computing the flattening coefficient $\zeta_{\boldsymbol{i}}$. We assume that there is a quantity corresponding to the pressure in gas dynamics (denoted here as p) which can act as a steepness indicator, and a quantity corresponding to the bulk modulus (denoted here as K, given as γp in a gas), that can be used to non-dimensionalize differences in p.

$$\begin{aligned}
\zeta_{\boldsymbol{v}} &= \begin{cases} \underset{0 \le d < \mathbf{D}}{min} \zeta_{\boldsymbol{v}}^d & \text{if } \sum_{d=0}^{\mathbf{D}-1} \Delta_1^d u_{\boldsymbol{v}}^d < 0 \\ 1 & \text{otherwise} \end{cases} \qquad (63) \\
\zeta_{\boldsymbol{v}}^d &= min_3(\widetilde{\zeta}^d, d)_{\boldsymbol{v}} \\
\widetilde{\zeta}_{\boldsymbol{v}}^d &= \eta(\Delta_1^d p_{\boldsymbol{v}}, \, \Delta_2^d p_{\boldsymbol{v}}, \, min_3(K, d)_{\boldsymbol{v}}) \\
\Delta_1^d p_{\boldsymbol{v}} &= \Delta^C p_{\boldsymbol{v}} \mid \Delta^L p_{\boldsymbol{v}} \mid \Delta^R p_{\boldsymbol{v}} \\
\Delta_2^d p_{\boldsymbol{v}} &= (\Delta_1^d p \ll \boldsymbol{e}^d)_{\boldsymbol{v}} + (\Delta_1^d p \ll -\boldsymbol{e}^d)_{\boldsymbol{v}} \mid 2\Delta_1^d p_{\boldsymbol{v}} \mid 2\Delta_1^d p_{\boldsymbol{v}}
\end{aligned}$$

The functions min_3 and ζ are given below.

$$\begin{aligned}
min_3(q, d)_{\boldsymbol{v}} &= \text{min}((q \ll \boldsymbol{e}^d)_{\boldsymbol{v}}, q_{\boldsymbol{v}}, (q \ll -\boldsymbol{e}^d)_{\boldsymbol{v}}) \mid \text{min} q_{\boldsymbol{v}}, (q \ll -\boldsymbol{e}^d)_{\boldsymbol{v}}) \mid \text{min}(q \ll \boldsymbol{e}^d)_{\boldsymbol{v}}, q_{\boldsymbol{v}}) \\
\zeta(\delta p_1, \delta p_2, p_0) &= \begin{cases} 0 & \text{if } \frac{|\delta p_1|}{p_0} > d \text{ and } \frac{|\delta p_1|}{|\delta p_2|} > r_1 \\ 1 - \frac{\frac{|\delta p_1|}{|\delta p_2|} - r_0}{r_1 - r_0} & \text{if } \frac{|\delta p_1|}{p_0} > d \text{ and } r_1 \ge \frac{|\delta p_1|}{|\delta p_2|} > r_0 \\ 1 & \text{otherwise} \end{cases} \\
r_0 &= 0.75, \;\; r_1 = 0.85, \;\; d = 0.33
\end{aligned} \qquad (64)$$

Note that min_3 is not the minimim over all available VoFs but involves the minimum of shifted VoFs which includes an averaging operation.

7 Computing fluxes at the irregular boundary

The flux at the embedded boundary is centered at the centroid of the boundary $\bar{\mathbf{x}}$. We extrapolate the primitive solution in space from the cell center. We then transform to the conservative solution and extrapolate in time using the stable, non-conservative estimate of the flux divergence described in equation 14.

$$W_{\boldsymbol{v},B} = W^n_{\boldsymbol{v}} + \sum_{d=0}^{D-1} (\bar{x}_d \Delta^d W^n_{\boldsymbol{v}}) \tag{65}$$

$$U^{n+\frac{1}{2}}_{\boldsymbol{v},B} = U(W_{\boldsymbol{v},B}) - \frac{\Delta t}{2} (D \cdot \vec{F})^{NC} \tag{66}$$

$$F^{n+\frac{1}{2}}_{\boldsymbol{v},B} = R_B(U^{n+\frac{1}{2}}_{\boldsymbol{v},B}, \boldsymbol{n}^B_{\boldsymbol{v}}) \tag{67}$$

If we are using solid-wall boundary condtions at the irregular boundary, we calculate an approximation of the divergence of the velocity at the irregular cell $D(\vec{u})_{\boldsymbol{v}}$ and use it to modify the flux to be consistent with artificial viscosity. The d-direction momentum flux at the irregular boundary is given by $-p^r n^d$ where p^r is the pressure to emerge from the Riemann solution in equation 67. For artificial viscosity, we modify this flux as follows.

$$(D\vec{u})_{\boldsymbol{v}} = \sum_{d'=0}^{D-1} \Delta^{d'} u^{d'}_{\boldsymbol{v}}$$

$$p^r = p^r - 2K_0 \ \mathsf{max}(-(D\vec{u})_{\boldsymbol{v}}, 0) \vec{u} \cdot \hat{n}$$

8 Results

We run the Modiano problem for one time step to compute the truncation error of the operator. The error at a given level of refinement E^h is approximated by

$$E^{\mathsf{trunc}} = \frac{U^h(t) - U^e(t)}{t} \tag{68}$$

where $U^h(t)$ is the discrete solution and $U^e(t)$ is the exact solution at time $t = \Delta t$. We run the Modiano problem for a fixed time to compute the solution error of the operator. The error at a given level of refinement E^h is approximated by

$$E^{\mathsf{soln}} = U^h(t) - U^e(t) \tag{69}$$

where $U^h(t)$ is the discrete solution and $U^e(t)$ is the exact solution at time t. The order of convergence p is given by

$$p = \frac{\log(\frac{|E^{2h}|}{|E^h|})}{\log(2)} \tag{70}$$

References

[CF48] R. Courant and K. O. Friedrichs. *Supersonic Flow and Shock Waves*. NYU, New York, NY, 1948.

Variable	Coarse Error	Fine Error	Order
mass-density	3.127796e-05	1.669137e-05	9.060445e-01
x-momentum	3.292329e-05	1.675957e-05	9.741235e-01
y-momentum	6.766401e-05	3.141857e-05	1.106771e+00
energy-density	1.094807e-04	5.842373e-05	9.060502e-01

Table 1: Truncation error convergence rates using L-0 norm. $h_f = \frac{1}{512}$ and $h_c = 2h_f$, $D = 2$

Variable	Coarse Error	Fine Error	Order
mass-density	7.358933e-08	1.616991e-08	2.186185e+00
x-momentum	7.569344e-08	2.010648e-08	1.912508e+00
y-momentum	1.764416e-07	4.648945e-08	1.924216e+00
energy-density	2.575709e-07	5.659651e-08	2.186185e+00

Table 2: Truncation error convergence rates using L-1 norm. $h_f = \frac{1}{512}$ and $h_c = 2h_f$, $D = 2$

Variable	Coarse Error	Fine Error	Order
mass-density	4.010155e-07	1.164273e-07	1.784228e+00
x-momentum	6.057493e-07	2.402295e-07	1.334308e+00
y-momentum	1.717569e-06	5.992271e-07	1.519193e+00
energy-density	1.403616e-06	4.075112e-07	1.784237e+00

Table 3: Truncation error convergence rates using L-2 norm. $h_f = \frac{1}{512}$ and $h_c = 2h_f$, $D = 2$

Variable	Coarse Error	Fine Error	Order
mass-density	3.769203e-07	7.212809e-08	2.385626e+00
x-momentum	3.427140e-07	7.681266e-08	2.157589e+00
y-momentum	7.501614e-07	1.692840e-07	2.147755e+00
energy-density	1.319233e-06	2.524508e-07	2.385625e+00

Table 4: Solution error convergence rates using L-0 norm. $h_f = \frac{1}{512}$ and $h_c = 2h_f$, $D = 2$

[CG85] P. Colella and H. M. Glaz. Efficient solution algorithms for the Riemann problem for real gases. *J. Comput. Phys.*, 59:264, 1985.

[CGL+00] P. Colella, D. T. Graves, T. J. Ligocki, D. F. Martin, D. Modiano, D. B. Serafini, and B. Van Straalen. Chombo Software Package for AMR Applications - Design Document. unpublished, 2000.

[Col90] Phillip Colella. Multidimensional upwind methods for hyperbolic conservation laws. *J. Comput. Phys.*, 87:171–200, 1990.

[Sal94] Jeff Saltzman. An unsplit 3d upwind method for hyperbolic conservation laws. *J. Comput. Phys.*, 115:153–168, 1994.

Variable	Coarse Error	Fine Error	Order
mass-density	1.103779e-09	1.855826e-10	2.572317e+00
x-momentum	1.125935e-09	2.356203e-10	2.256588e+00
y-momentum	1.617258e-09	2.371548e-10	2.769649e+00
energy-density	3.863314e-09	6.495531e-10	2.572320e+00

Table 5: Solution error convergence rates using L-1 norm. $h_f = \frac{1}{512}$ and $h_c = 2h_f$, $D = 2$

Variable	Coarse Error	Fine Error	Order
mass-density	5.553216e-09	1.114919e-09	2.316385e+00
x-momentum	6.038922e-09	1.251264e-09	2.270905e+00
y-momentum	9.515687e-09	2.244841e-09	2.083695e+00
energy-density	1.943688e-08	3.902358e-09	2.316379e+00

Table 6: Solution error convergence rates using L-2 norm. $h_f = \frac{1}{512}$ and $h_c = 2h_f$, $D = 2$